JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 20, No. 4, July–August 1997

Mixed H_2/H_∞ Method Suitable for Gain Scheduled Aircraft Control

Shyh-Pyng Shue,* M. E. Sawan,† and Kamran Rokhsaz‡
Wichita State University, Wichita, Kansas 67260-0044

The formulations of mixed H_2/H_∞ gain scheduling control of aircraft in longitudinal motion are developed. The coefficients of a linear model of aircraft are estimated from the aircraft geometry. Based on flight Mach number, gain scheduling is applied with the mixed H_2/H_∞. It is shown that the state equation of aircraft can be mathematically simplified to the standard mixed H_2/H_∞ problem. The advantage of decoupling the aircraft system based on phugoid (slow) and short period (fast) modes is to allow the slow model with its disturbance to be controlled by the H_2 method, whereas the fast system with the fast disturbance is to be stabilized by the H_∞ technique. A comparison of this method with the standard H_∞ control is made using aircraft coefficients derived for a large commercial airplane. It is shown that the current method will provide better performance for the given aircraft while the condition on the disturbance attenuation is also satisfied. The actual model controlled by the estimated controller and by the real controller is also discussed. It is shown that the estimated controller performs as well as the controller derived from the actual model.

Nomenclature

A	= state matrix of the aircraft system
B	= control matrix
b_w, b_H	= span of wing and horizontal tail
C_{D_0}	= drag coefficient on zero angle of attack (AOA)
C_{D_q}, C_{D_u}, C_{D_α}, $C_{D_{\dot\alpha}}$	= variation of drag coefficient with respect to (w.r.t.) q, u, α, $\dot\alpha$
C_{L_0}	= lift coefficient at zero AOA
C_{L_q}, C_{L_u}, C_{L_α}, $C_{L_{\dot\alpha}}$	= variation of lift coefficient w.r.t. q, u, α, $\dot\alpha$
C_{m_0}	= pitching moment coefficient for zero AOA
C_{m_q}, C_{m_u}, C_{m_α}, $C_{m_{\dot\alpha}}$	= variation of pitching moment coefficient w.r.t. q, u, α, $\dot\alpha$
C_w, C_H	= chord of wing and horizontal tail
C_1, C_2, C_3	= output matrices
D_{13}, D_{23}, D_{31}	= control matrices in the output
$\boldsymbol{d}_1$	= type of noise, $[u_g \;\; q_g \;\; \alpha_g]$
$\boldsymbol{d}_2$	= output noise
e	= Oswald efficiency factor
G_1, G_2	= exogenous input matrix
g	= acceleration of gravity
H_∞, J_∞	= Riccati domains for the H_∞ method
I_{yy}	= moment of inertia about the Y axis
I_2	= identity matrix (2×2)
i, j, l, n	= integer
J	= performance index
K_s, K_f	= control gain before computing composite control
L_s, L_f	= filter gains
M_s, M_f	= control gains for composite control
M_q, M_u, M_α, $M_{\dot\alpha}$, M_{δ_e}, M_θ	= dimensional variation of pitching moment w.r.t. q, u, α, $\dot\alpha$, δ_e, θ
M_∞	= Mach number
P_s, Q_s	= Riccati solution for the slow mode
q	= pitch rate
q_g	= disturbance in pitch rate
$\hat{q}$	= dynamic pressure
R_2, R_3	= weight matrices
$T_{z_2 d_2}$	= transfer function from d_2 to z_2
U_0	= steady-state airspeed
u	= horizontal velocity
u_g	= disturbance in the horizontal velocity
V_H	= S_H / S_w
V_s	= intensity of slow gust
V_u	= intensity of gust of the horizontal velocity
W	= weight of aircraft
$\boldsymbol{w}$	= system disturbance
X_{cg}, X_{ac_w}, X_{ac_H}	= locations of c.g. and A.C. w.r.t. wing and tail, respectively; dimensional variation of horizontal force w.r.t. u, α, θ, and q, respectively
X_∞, Y_∞	= Riccati solutions for the H_∞ method
$\boldsymbol{x}$	= system state vector, $[u \;\; \theta \;\; \alpha \;\; q]^T$
$\boldsymbol{x}_0$	= initial condition of the system response
$\boldsymbol{y}$	= measurement output
Z_q, Z_α, Z_{δ_e}, Z_θ	= dimensional variation of vertical force w.r.t. q, α, δ_e, θ
z_1, z_2	= performance output
α	= AOA
β	= $\sqrt{(1 - M_\infty^2)}$
γ	= flight climb angle
γ_0	= steady-state climb angle
γ^*	= minimum value of the disturbance attenuation
δ_e	= control input (elevator)
ε	= singular value
$\boldsymbol{\zeta}$	= estimated fast state
η_H	= dynamic pressure ratio at the horizontal tail, $\hat{q}_H / \hat{q}_w$
θ	= pitch angle
κ	= ratio of actual lift curve slope to 2π
Λ	= sweep angle
λ	= taper ratio
λ_s, λ_f, λ_c, λ_o	= eigenvalues of the slow and fast modes and closed-loop and open-loop systems
Ξ	= $1 - (Z_{\dot\alpha} / U_0)$
$\boldsymbol{\xi}$	= estimated slow state
υ	= fast mode time scale

Subscripts

f	= fast mode
g	= coefficient for gust
r	= coefficient for the real model
s	= slow mode
y, z	= output

Received July 16, 1996; revision received Feb. 6, 1997; accepted for publication Feb. 6, 1997.

*Engineering Doctoral Fellow, Department of Aerospace Engineering. Student Member AIAA.
†Professor, Department of Electrical Engineering.
‡Assistant Professor, Department of Aerospace Engineering. Associate Fellow AIAA.

I. Introduction

OPTIMAL control methods have been studied extensively since the matrix differential Riccati equation was found for the optimal control system in Ref. 1. Thereafter, the Kalman filter theory was well developed.[2] Combining both methods applies to systems having the full state vector available with active noise, which is assumed to be white, zero mean, and Gaussian. This method is known as the linear quadratic Gaussian (LQG) method, also named the H_2 method.[3,4] In this technique, a quadratic performance index is minimized to find the associated optimal controller. The LQG method provides excellent performance because the error between the state vector and the estimated state variable is always asymptotically stable, while minimizing a quadratic energy function.[3]

The formulation of the H_∞ method for the stochastic linear system appeared in the early 1980s.[5] One of the first texts[6] related to the H_∞ method was published in 1987. The H_∞ method using state feedback was further developed in Refs. 7 and 8. In these two papers, the H_∞ control of a stochastic system using state feedback is shown to require satisfying an algebraic Riccati equation. The H_∞ method using output feedback was presented in Ref. 9, in which two Riccati equations were developed and the necessary conditions for robustness were also specified.

In the H_∞ method, three types of noise are considered: the command error, the system disturbance, and the output error. All three disturbances are considered as the exogenous inputs. As long as the Euclidean norm of the transfer function from the performance output to the system disturbance, denoted as $\|T_{zw}\|$, is minimized, the system is stabilized and will be robustly controlled.[6] The minimum value of this norm results in the closed-loop poles being very close to the imaginary axis. Using the minimum value will result in sacrificing the performance in favor of robustness. The mixed H_2/H_∞ method helps to alleviate this problem. Several of the most important works in this area are in Refs. 10–12. In Ref. 11, it is shown that the stochastic system can be robustly controlled by the H_2/H_∞ performance even if the different types of noise are causally dependent. Reference 12 showed that a system of this type can be optimally controlled by the mixed H_2/H_∞ controller. Based on this, H_2, H_∞, and mixed H_2/H_∞ controllers have been extensively applied to aircraft control.[13–16]

Gain scheduling control of an aircraft is necessary because the aircraft stability can vary significantly depending on the flight regime. The stability and control derivatives of an aircraft are strongly affected by the airspeed, air density, location of the center of gravity, etc. In addition, gust standard deviation is also dependent on altitude. References 17–20 contain examples of gain scheduling and H_∞ optimal control of aircraft and missiles. Usually, such gain scheduling is made dependent on the airspeed. A small range of altitudes is assumed so that the changes of the coefficients of aircraft are bounded. This allows the system to be controlled by the H_∞ optimal gain scheduling controller. Furthermore, an aircraft model has been shown to be a singularly perturbed system both in longitudinal and lateral motions.[21] The disturbances in the state equation of an aircraft model can be classified as fast and slow disturbances, if the model is separated into two subsystems using the phugoid and short period modes. Consequently, the aircraft can be controlled by a composite controller using the H_∞ method.[22] The H_∞ optimal control for singularly perturbed systems can be achieved in two situations: one is called the perfect state measurement,[23] and the other is called the imperfect state measurement.[24] Reducing the aircraft system to two subsystems allows the design to be focused on each of the reduced subsystems. In this fashion, the system will have two penalty performances, one for each subsystem. This allows the slow mode and fast mode to be controlled separately. Because of this arrangement, the system performance is improved substantially, and the disturbance attenuation of the system for the H_∞ norm is also satisfied.

For a fixed altitude and c.g. location, the coefficients of the linear aircraft depend on the geometry and Mach number. For subsonic Mach number, i.e., less than 0.75, these coefficients can be estimated using the expressions given in Ref. 25. The differences between the estimated and the actual model constitute the bounded uncertainties built into the H_∞ controller. Therefore, the estimated controller computed by the mixed H_2/H_∞ method can replace that based on the aircraft system without any degradation in performance.

Concepts from all of the cited methods, which are the H_2, H_∞, mixed H_2/H_∞, singular perturbation, gain scheduling, and estimated aircraft coefficient, will be used for designing the mixed H_2/H_∞ gain scheduling composite controller for an aircraft longitudinal motion. The linear model used will be based on the estimated aircraft coefficients. It will be shown that the slow phugoid mode and the fast short period mode will be controlled by the H_2 and the H_∞ techniques, respectively. Unlike the standard H_2/H_∞ problem, the current method creates a mixed controller resulting in better performance while satisfying the disturbance condition. The gain scheduling employed here is based on Mach number.

The linear model of a large commercial airplane[26] is employed to illustrate the theory. Also at discrete Mach numbers, comparisons are provided with the pure H_∞ method and with the controller from the actual model.

II. Problem Statement

Consider the longitudinal motion of an aircraft linear model as follows[27]:

$$\dot\theta = q, \qquad \dot q = \tilde M_u u + \tilde M_\alpha \alpha + \tilde M_q q + \tilde M_\theta \theta + \tilde M_{\delta_e}\delta_e$$
$$\dot u = X_u u + X_\alpha \alpha - g\cos\gamma_0 \theta + X_{\delta_e}\delta_e \tag{1}$$
$$\dot\alpha = Z_u^* u + Z_\alpha^* \alpha + Z_q^* q + Z_{\delta_e}^*\delta_e$$

where

$$\tilde M_u = (M_u + M_{\dot w} Z_u), \qquad \tilde M_\alpha = (M_w + M_{\dot w} Z_w)U_0$$
$$\tilde M_q = (M_q + U_0 M_{\dot w}), \qquad \tilde M_\theta = -g M_{\dot w}\sin\gamma_0$$
$$\tilde M_{\delta_e} = \left(M_{\delta_e} + M_{\dot w} Z_{\delta_e}\right), \qquad Z_\theta^* = -\frac{g\sin\gamma_0}{\Xi}$$
$$Z_q^* = \frac{Z_q + U_0}{\Xi}, \qquad Z_u^* = \frac{Z_u}{U_0}, \qquad Z_\alpha^* = Z_\alpha U_0$$
$$Z_{\delta_e}^* = \frac{Z_{\delta_e}}{U_0}, \qquad \Xi = 1 - \frac{Z_{\dot\alpha}}{U_0}$$

From Refs. 26–28, the effect of gusts on the longitudinal motion of the aircraft can be given as follows:

$$u = u + u_g, \qquad q = q + q_g, \qquad \alpha = \alpha + \alpha_g \tag{2}$$

Note that the disturbances in the pitch angle is indirectly computed from the pitch rate. From Ref. 27, the performance output variable of interest is

$$Z_1 = a_{z_1} = (Z_u - l_x \tilde M_u)u + (Z_w - l_x \tilde M_w)U_0\alpha$$
$$- l_x \tilde M_q q + \left(Z_{\delta_e} - l_x \tilde M_{\delta_e}\right)\delta_e \tag{3}$$

The measured output is

$$\boldsymbol{y} = \gamma = \theta - \alpha + d_2 \tag{4}$$

In this case, the disturbance $d_1 = [u_g \;\; q_g \;\; \alpha_g]^T$ and the output noise d_2 are independent. Knowing that the phugoid mode is predominantly oscillatory in u and θ, while the short period mode is composed of variation in q and α, Eqs. (1–4) can be arranged as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B_1\boldsymbol{d}_1 + B_2\delta_e, \qquad \boldsymbol{z}_1 = C_1\boldsymbol{x} + D_{11}\boldsymbol{d}_1 + D_{12}\delta_e$$
$$\boldsymbol{y} = C_2 x + D_{21} d_2 + D_{22}\delta_e \tag{5}$$

with

$$A = \begin{bmatrix} X_u & -g\cos\gamma_0 & X_\alpha/U_0 & 0 \\ 0 & 0 & 0 & 1 \\ Z_u^* & Z_\theta^* & Z_\alpha^* & Z_q^* \\ \tilde{M}_u & \tilde{M}_\theta & \tilde{M}_\alpha & \tilde{M}_q \end{bmatrix} \tag{6}$$

$$B_1 = \begin{bmatrix} X_u & X_\alpha/U_0 & 0 \\ 0 & 0 & 1 \\ Z_u^* & Z_\alpha^* & Z_q^* \\ \tilde{M}_u & \tilde{M}_\alpha & \tilde{M}_q \end{bmatrix}, \qquad B_2 = \begin{bmatrix} X_{\delta_e} \\ 0 \\ Z_{\delta_e}^* \\ \tilde{M}_{\delta_e} \end{bmatrix} \tag{7}$$

$$C_1 = [(Z_u - l_x\tilde{M}_u) \quad 0 \quad (Z_w - l_x\tilde{M}_w) \quad -l_x\tilde{M}_q]$$
$$C_2 = [0 \quad 1 \quad -1 \quad 0] \tag{8}$$

$$D_{12} = \left(Z_{\delta_e} - l_x\tilde{M}_{\delta_e}\right), \qquad D_{21} = 1 \tag{9}$$

Note that D_{11} and D_{22} are zeros here. This implies that the assumptions of the standard H_2/H_∞ problem in Ref. 9 are reasonable. Furthermore, $D_{12}^T D_{12}$ and $D_{21}D_{21}^T$ are nonsingular. Therefore, the transfer function of the system can be written as

$$G(s) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \tag{10}$$

Note that the form of this transfer function is very close to that of Ref. 9. Therefore, the following assumptions are needed for the system to result in a feasible solution: 1) (A, B_1) and (A, B_2) are stabilizable, 2) (A, C_1) and (A, C_2) are detectable, 3) $D_{12}^T D_{12} = R_1 > 0$, and 4) $D_{21}^T D_{21} = R_2 > 0$.

III. Methodology

In this section, the aircraft system is decoupled using the phugoid and short period modes. The H_2 method is applied for controlling the phugoid system with the disturbance u_g, and the H_∞ technique is used for stabilizing the short period model with the disturbances q_g and α_g. The composite controller is computed from these two gains. Note that because of large differences between the two modes, neither the standard H_∞ nor the standard mixed H_2/H_∞ can control the system with reasonable performance. However, the arrangement results in better performance because a performance index is minimized while satisfying the disturbance attenuation.

A. Singular Perturbation Method

From the aircraft model shown in Eq. (5), the following singularly perturbed system can be established:

$$\dot{\boldsymbol{x}}_1 = A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 + B_{11}\boldsymbol{w} + B_{21}\delta_e$$
$$\varepsilon\dot{\boldsymbol{x}}_2 = A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 + B_{12}\boldsymbol{w} + B_{22}\delta_e \tag{11}$$
$$z_1 = C_{11}\boldsymbol{x}_1 + C_{12}\boldsymbol{x}_2 + D_{12}\delta_e, \qquad \boldsymbol{y} = C_{21}\boldsymbol{x}_s + C_{22}\boldsymbol{x}_2 + D_{21}\boldsymbol{d}$$

where $x_1 = [u, \theta]$, $x_2 = [\alpha, q]$, and ε is a small positive parameter calculated from the slow and fast eigenvalues defined as follows:

$$\varepsilon = \frac{\sqrt[i]{\left|\prod_{n=1}^{i}\lambda_{s_n}\right|}}{\sqrt[j]{\left|\prod_{l=1}^{j}\lambda_{f_l}\right|}} \tag{12}$$

The method of computing the singular value ε is useful for any systems having eigenvalues that are different by orders of magnitude. Contrary to Ref. 22, this method can be used even for a system with pure real eigenvalues for the fast mode. Note that the coefficient matrices of Eq. (11) now become

$$A_{11} = \begin{bmatrix} X_u & -g\cos\gamma_0 \\ 0 & 0 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} X_\alpha/U_0 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

$$A_{21} = \varepsilon\begin{bmatrix} Z_u^* & Z_\theta^* \\ \tilde{M}_u & \tilde{M}_\theta \end{bmatrix}, \qquad A_{22} = \varepsilon\begin{bmatrix} Z_\alpha^* & Z_q^* \\ \tilde{M}_\alpha & \tilde{M}_q \end{bmatrix} \tag{14}$$

$$B_{11} = \begin{bmatrix} X_u & X_\alpha/U_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_{12} = \varepsilon\begin{bmatrix} Z_u^* & Z_\alpha^* & Z_q^* \\ \tilde{M}_u & \tilde{M}_\alpha & \tilde{M}_q \end{bmatrix} \tag{15}$$

$$B_{21} = \begin{bmatrix} X_{\delta_e} \\ 0 \end{bmatrix}, \qquad B_{22} = \varepsilon\begin{bmatrix} Z_{\delta_e}^* \\ \tilde{M}_{\delta_e} \end{bmatrix} \tag{16}$$

$$C_{11} = [(Z_u - l_x\tilde{M}_u) \quad 0], \qquad C_{12} = [(Z_w - l_x\tilde{M}_w) \quad -l_x\tilde{M}_q]$$
$$C_{21} = [0 \quad 1], \qquad C_{22} = [1 \quad 0] \tag{17}$$

Note that D_{12} and D_{21} are unchanged.

B. Mixed H_2/H_∞ Composite Controller

1. H_2 Method for the Slow Mode

To obtain the slow mode, setting $\varepsilon = 0$, x_2 can be determined (denoted as x_{2_s}) as follows:

$$\boldsymbol{x}_{2_s} = -A_{22}^{-1}\left(A_{21}\boldsymbol{x}_s + B_{12}\boldsymbol{w}_s + B_{22}\delta_{e_s}\right) \tag{18}$$

The slow subsystem can be obtained by substituting Eq. (18) back into Eq. (11) as follows:

$$G_s(s) = \begin{cases} \dot{\boldsymbol{x}}_s = A_0\boldsymbol{x}_s + G_0\boldsymbol{w}_s + B_0\delta_{e_s} \\ z_s = C_s\boldsymbol{x}_s + D_s\delta_{e_s} \\ \boldsymbol{y}_s = C_0\boldsymbol{x}_s + D_0\boldsymbol{d}_s + E_0 w_s \end{cases} \tag{19}$$

where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \qquad B_0 = B_{21} - A_{12}A_{22}^{-1}B_{22}$$
$$G_0 = B_{11} - A_{12}A_{22}^{-1}B_{12} \tag{20}$$

$$C_s = C_{11} - C_{12}A_{22}^{-1}A_{21}, \qquad D_s = D_{12} - C_{12}A_{22}^{-1}B_{22} \tag{21}$$

$$C_0 = C_{21} - C_{22}A_{22}^{-1}A_{21}, \qquad D_0 = D_{21} - C_{22}A_{22}^{-1}B_{21}$$
$$E_0 = -C_{22}A_{22}^{-1}B_{21} \tag{22}$$

Note that the slow subsystem may contain a noise term in the control output z_s and a control input term in the measured output y_s. As in Ref. 29, these terms are assumed to be zero. In addition, the slow filter is shown as follows:

$$H(s) = \left\{\dot{\boldsymbol{\xi}} = A_0\boldsymbol{\xi} + B_0\delta_{e_s} + L_s(\boldsymbol{y}_s - C_0\boldsymbol{\xi})\right. \tag{23}$$

where L_s is the filter gain. The objective of this method is to control the slow mode by the H_2 method by minimizing the following cost function:

$$J_s = E\left[\frac{1}{2}\int_0^\infty \left\{\boldsymbol{x}_s^T C_s^T C_s\boldsymbol{x}_s + \delta_{e_s}^T D_s^T D_s\delta_{e_s}\right\}dt\right] \tag{24}$$

where $E[\,]$ denotes the expected value of the interpolated function. Note that $D_s^T D_s$ is invertible and positive. The optimal controller can be found based on the given cost function as

$$\delta_{e_s} = -K_s\boldsymbol{\xi} = -\left(D_s^T D_s\right)^{-1}B_0^T P_s\boldsymbol{\xi} \tag{25}$$

where P_s is the solution of the following algebraic Riccati:

$$A_0^T P_s + P_s A_0 + C_s^T C_s - P_s B_0\left(D_s^T D_s\right)^{-1}B_0^T P_s = 0 \tag{26}$$

The estimated slow state ξ is computed from the slow filter design (23), which depends on the intensity of the disturbance. Assume that the intensity of the original state disturbance can also be separated into a slow and a fast mode. This assumption results in

$$V_s = V_u \tag{27}$$

The noise densities of the slow mode for the measured output is assumed to be given by the identity matrix.

The slow Kalman filter design is used to find a Kalman filter gain L_s, such that the following Kalman observer is always stable:

$$\dot{\boldsymbol{\xi}} = A_0\boldsymbol{\xi} + B_0\delta_{e_s} + L_s(\boldsymbol{y}_s - C_0\boldsymbol{\xi}) \tag{28}$$

where

$$L_s = \left(Q_s C_0^T + G_0 E_0^T\right)(V_l)^{-1} \tag{29}$$

Note that Q_s is the solution of the following Riccati equation:

$$A_0 Q_s + Q_s A_0^T + V_u - Q_s C_0^T V_l^{-1} C_0 Q_s = 0 \tag{30}$$

2. H_∞ Technique for the Fast Mode

Setting $\boldsymbol{x}_f = \boldsymbol{x}_2 - \boldsymbol{x}_{2_s}$, $\delta_{e_f} = \delta_e - \delta_{e_s}$, $\boldsymbol{w}_f = \boldsymbol{w} - \boldsymbol{w}_s$, and $\boldsymbol{y}_f = \boldsymbol{y} - \boldsymbol{y}_s$, then the fast mode can be obtained as

$$G_f(s) = \begin{cases} \varepsilon\dot{\boldsymbol{x}}_f = A_{22}\boldsymbol{x}_f + B_{12}\boldsymbol{w}_f + B_{22}\delta_{e_f} \\ \boldsymbol{z}_f = C_f\boldsymbol{x}_f + D_f\delta_{e_f} \\ \boldsymbol{y}_f = C_y\boldsymbol{x}_f + D_y\boldsymbol{d} \end{cases} \tag{31}$$

Here the fast mode time derivative is with respect to $\upsilon = t' - t/\varepsilon$ where t' varies at the same rate as t/ε. Note that t is considered to be frozen for the fast mode. The H_∞ filter for the fast mode is designed as

$$F(s) = \left\{\dot{\boldsymbol{\zeta}} = A_{22}\boldsymbol{\zeta} + B_{22}\delta_{e_f} + L_f(\boldsymbol{y}_f - C_y\boldsymbol{\zeta})\right. \tag{32}$$

where L_f is the H_∞ filter gain.

The disturbance attenuation γ^* of the H_∞ control for the fast mode is prescribed and is computed from the full model. To ensure the controlled system is stable, the value of γ^* is chosen very conservatively, making the controller suboptimal. Using this value, the Riccati solutions for the fast mode are associated with the following Hamiltonian matrices: $X_\infty \in \mathrm{dom}(H_\infty)$, where

$$H_\infty = \begin{bmatrix} A_{22} & \gamma^{*^{-2}} B_{12}\left(D_f^T D_f\right)^{-1} B_{12}^T - B_{22}\left(D_f^T D_f\right)^{-1} B_{22}^T \\ -C_f^T C_f & -A_{22}^T \end{bmatrix} \tag{33}$$

and $Y_\infty \in \mathrm{dom}(J_\infty)$, where

$$J_\infty = \begin{bmatrix} A_{22}^T & \gamma^{*^{-2}} C_f^T\left(D_y D_y^T\right)^{-1} C_f - C_y^T\left(D_y D_y^T\right)^{-1} C_y \\ -B_{12}B_{12}^T & -A_{22} \end{bmatrix} \tag{34}$$

The Riccati equations corresponding to the preceding two Hamiltonian matrices are described as follows:

$$X_\infty A_{22} + A_{22}^T X_\infty + C_f^T C_f + X_\infty \times \left[\gamma^{-2} B_{12}\left(D_f^T D_f\right)^{-1} B_{12}^T - B_{22}\left(D_f^T D_f\right)^{-1} B_{22}^T\right] X_\infty = 0 \tag{35}$$

$$Y_\infty A_{22}^T + A_{22} Y_\infty + B_{12}B_{12}^T + Y_\infty \times \left[\gamma^{-2} C_f^T\left(D_y D_y^T\right)^{-1} C_f - C_y^T\left(D_y D_y^T\right)^{-1} C_y\right] Y_\infty = 0 \tag{36}$$

Note that γ^* must be selected to satisfy the following constraint:

$$\rho(X_\infty Y_\infty) < \gamma^{*^2} \tag{37}$$

Therefore, the H_∞ controller and the exogenous input for the fast mode are

$$\begin{aligned} \delta_{e_f} &= -K_f\boldsymbol{\zeta} = -\left(D_f^T D_f\right)^{-1} B_{22}^T X_\infty \boldsymbol{\zeta} \\ \boldsymbol{w}_f &= -\left(D_f^T D_f\right)^{-1} B_{12}^T X_\infty \boldsymbol{\zeta} \end{aligned} \tag{38}$$

where the estimated state $\boldsymbol{\zeta}$ for the fast system is calculated from the H_∞ filter described in Eq. (32). Furthermore, the H_∞ filter gain of Eq. (32) is

$$L_f = -Z_\infty Y_\infty C_y^T\left(D_y D_y^T\right)^{-1} \tag{39}$$

where

$$Z_\infty = \left(I - \gamma^{*^{-2}} X_\infty Y_\infty\right)^{-1} \tag{40}$$

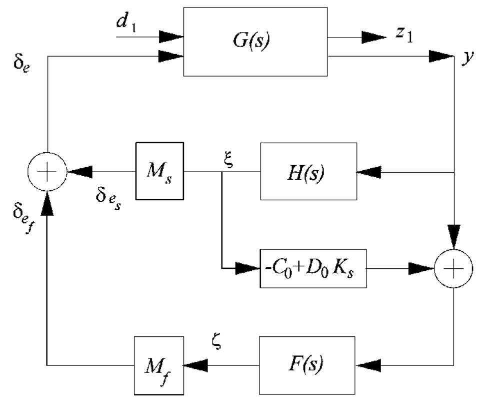

Fig. 1 Block diagram for the mixed H_2/H_∞ composite controller.

3. Composite Controller

Because the preceding formulation is for the two independent subsystems, it is necessary to combine both controllers as follows:

$$\delta_e = \delta_{e_s} + \delta_{e_f} = -K_s\boldsymbol{\xi} - K_f\boldsymbol{\zeta} \tag{41}$$

From Ref. 30, the composite controller is

$$\delta_e = M_s\boldsymbol{x}_s + M_f\boldsymbol{x}_f \tag{42}$$

where

$$M_s = -\left(I_2 - K_f A_{22}^{-1} B_{22}\right)K_s - K_f A_{22}^{-1} A_{21} \tag{43}$$

$$M_f = -K_f \tag{44}$$

The control diagram for the composite controller is shown in Fig. 1.

C. Gain Scheduling Based on Mach Number

The equations used for estimating the aerodynamic coefficients and derivatives of the aircraft are given in the Appendix. The values of these parameters change with Mach number, changing the control gains. Only subsonic flight is considered, and the Mach numbers are varied from $M_\infty = 0.45$ to 0.75. These flight conditions are assumed to be steady state and are used for gain scheduling. As shown in Ref. 26, the aerodynamic coefficients of the aircraft change slightly in this range of Mach numbers. Altitude is assumed to be constant, and only longitudinal motion is considered. Therefore, the linear model of the aircraft is assumed to depend only on Mach number. Consequently, every coefficient in Eqs. (9) and (31) varies with Mach number. This results in variations in the controller gains K_s and K_f and, therefore, implicit dependence of the controller δ_e on flight Mach number. Furthermore, inasmuch as the aerodynamic coefficients change slightly in the subsonic flight regime, the uncertainties in the system coefficients remain bounded. Note that determination of the controller is independent of the real system as long as Mach number is known. Recall that the H_∞ method allows for three types of noise: the command error, the system disturbance, and the measurement output error. The bounded uncertainties of the aerodynamic coefficients can be interpreted as command error.[31,32] As long as these uncertainties are bounded, the system can always be robustly controlled.

In the next section, it is shown that a mixed H_2/H_∞ gain scheduling controller, designed based on the estimated aircraft coefficients has excellent performance. Furthermore, it is shown that the same controller can also control the actual system without uncertainties in its aerodynamic coefficients. There is an example in the next section to illustrate the theory.

IV. Results and Comparisons

A. Illustration

In this section, a large commercial airplane is used to illustrate the theory. The detailed geometric characteristics of this aircraft can be found in Refs. 26 and 33. Table 1 shows the inertial and the atmospheric conditions used in the current study.[26,34] Based on these data, the state equations for the longitudinal motion at different Mach numbers can be estimated as follows.

Table 1 General information for the airplane[34]

Variable	Values
Weight	160,000 lb
Inertia	3.0×10^6 slug/ft^2
Air density	0.001268 slug/ft^3
C_{D0}	0.017
C_{m0}	0.0
C_{mT_0}	0.0
Wing mean geometric chord	16.52 ft
Steady-state initial altitude	2.5 deg
Speed of sound at 20,000 ft	1,030 ft/s
Oswald's efficiency factor	0.9
$\Lambda_{c/2}$	28 deg
Length overall	153.17 ft

1. $M_\infty = 0.75$

$$A_{M_\infty=0.75} = \begin{bmatrix} -0.0117 & -32.1694 & 0.01874 & 0 \\ 0 & 0 & 0 & 1 \\ -0.1351 & -1.3847 & -0.9046 & 754.4429 \\ 1.1874\times10^{-6} & 1.2175\times10^{-5} & -0.0060 & -0.6888 \end{bmatrix}$$

$$B_{1M_\infty=0.75} = \begin{bmatrix} -0.0117 & 0.0187 & 0 \\ 0 & 0 & 1 \\ -0.1350 & -0.9046 & 754.4429 \\ 1.1874\times10^{-6} & -0.0060 & -0.6888 \end{bmatrix}$$

$$B_{2M_\infty=0.75} = \begin{bmatrix} 0 \\ 0 \\ -52.2618 \\ -4.5570 \end{bmatrix}$$

$$C_{1M_\infty=0.75} = [-0.1370 \quad 0 \quad -0.9175 \quad -3.4110]$$

$$C_{2M_\infty=0.75} = [0 \quad -1 \quad 1 \quad 0]$$

$$D_{12M_\infty=0.75} = -30.2252, \qquad D_{21M_\infty=0.75} = 1$$

2. $M_\infty = 0.65$

$$A_{M_\infty=0.65} = \begin{bmatrix} -0.0119 & -32.1694 & 0.0232 & 0 \\ 0 & 0 & 0 & 1 \\ -0.1295 & -1.3847 & -0.7374 & 654.1081 \\ 1.0908\times10^{-6} & 1.1659\times10^{-5} & -0.0049 & -0.5716 \end{bmatrix}$$

$$B_{1M_\infty=0.65} = \begin{bmatrix} -0.0119 & 0.0232 & 0 \\ 0 & 0 & 1 \\ -0.1295 & -0.7374 & -654.1081 \\ 1.0975\times10^{-6} & -0.0049 & -0.5716 \end{bmatrix}$$

$$B_{2M_\infty=0.65} = \begin{bmatrix} 0 \\ 0 \\ -39.2544 \\ -3.5244 \end{bmatrix}$$

$$C_{1M_\infty=0.65} = [-0.1314 \quad 0 \quad -0.7479 \quad -2.8309]$$

$$C_{2M_\infty=0.65} = [0 \quad -1 \quad 1 \quad 0]$$

$$D_{12M_\infty=0.65} = -22.1944, \qquad D_{21M_\infty=0.65} = 1$$

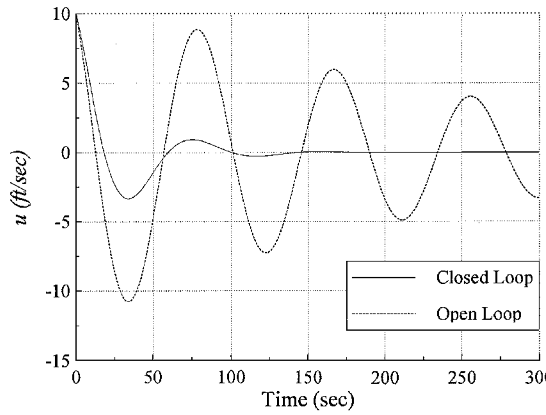

Fig. 2 Initial response of the open-loop and closed-loop horizontal airspeed for M_∞ = 0.75.

3. $M_\infty = 0.55$

$$A_{M_\infty=0.55} = \begin{bmatrix} -0.01291 & -32.1694 & 0.0287 & 0 \\ 0 & 0 & 0 & 1 \\ -0.1364 & -1.3847 & -0.5952 & 553.6394 \\ 1.1104\times10^{-6} & 1.2726\times10^{-5} & -0.0040 & -0.4677 \end{bmatrix}$$

$$B_{1M_\infty=0.55} = \begin{bmatrix} -0.0129 & 0.0287 & 0 \\ 0 & 0 & 1 \\ -0.1364 & -0.5952 & -553.6394 \\ 1.1104\times10^{-6} & -0.0040 & -0.4677 \end{bmatrix}$$

$$B_{2M_\infty=0.55} = \begin{bmatrix} 0 \\ 0 \\ -28.1052 \\ -2.5962 \end{bmatrix}$$

$$C_{1M_\infty=0.55} = [-0.1384 \quad 0 \quad -0.6037 \quad -2.3160]$$

$$C_{2M_\infty=0.55} = [0 \quad -1 \quad 1 \quad 0]$$

$$D_{12M_\infty=0.55} = -15.5270, \qquad D_{21M_\infty=0.55} = 1$$

The stability and control derivatives used for these equations were estimated using the methods of Refs. 25, 26, 28, and 36, as outlined in the Appendix. Although these equations are computed for three discrete Mach numbers, in practice, they will change continually depending on the flight Mach number.

For $M_\infty = 0.75$, the initial response of the open-loop system without any disturbance and control is compared with that of the closed-loop system in Fig. 2. The initial conditions imposed in this case were

$$\boldsymbol{x}_0 = [10\,\text{ft/s} \quad 0 \quad 0 \quad 5\,\text{deg}]^T$$

As seen in Fig. 2, the horizontal airspeed of the open-loop system oscillates for more than 500 s. However, the closed-loop horizontal velocity is damped out in approximately 150 s. Comparison of the two responses also shows that the frequency of this parameter remains unaffected by this control scheme. Therefore, effectively, the H_2/H_∞ composite controller has only changed the real parts of the system eigenvalues. Figure 3 shows the time history of angle of attack and climb angle for the open-loop and closed-loop systems. It is obvious that the frequency of the response has changed minimally, while the controlled response was damped out slightly faster. Both the open-loop and the closed-loop eigenvalues corresponding to the preceding cases are

$$\lambda_c = \begin{cases} -1.2918 \pm 2.4680i \\ -0.0306 \pm 0.0760i \end{cases} \quad \text{and} \quad \lambda_o = \begin{cases} -0.8040 \pm 2.1454i \\ -0.0044 \pm 0.0709i \end{cases}$$

Table 2 Solutions of the composite control gain and slow and fast filter gains

	$M_\infty = 0.45$	$M_\infty = 0.55$	$M_\infty = 0.65$	$M_\infty = 0.75$
M_s	$[-0.00023 \quad 0.2235]$	$[-0.00016 \quad 0.1591]$	$[-0.00012 \quad 0.123]$	$[-0.000098 \quad 0.1035]$
M_f	$[0.0005 \quad 0.3241]$	$[0.00044 \quad 0.2754]$	$[0.0004 \quad 0.2431]$	$[0.00038 \quad 0.2211]$
L_s	$[-0.9955 \quad 8.0780]^T$	$[-0.9964 \quad 8.0805]^T$	$[-0.9967 \quad 8.0780]^T$	$[-0.9968 \quad 8.0789]^T$
L_f	$[-0.0333 \quad -0.3320]^T$	$[-0.0312 \quad -0.5688]^T$	$[-0.0325 \quad -1.1561]^T$	$[-0.0426 \quad -4.0613]^T$
K_s	$[-0.00011 \quad 0.1386]$	$[-0.000078 \quad 0.1129]$	$[-0.000059 \quad 0.0873]$	$[-0.00005 \quad 0.0734]$

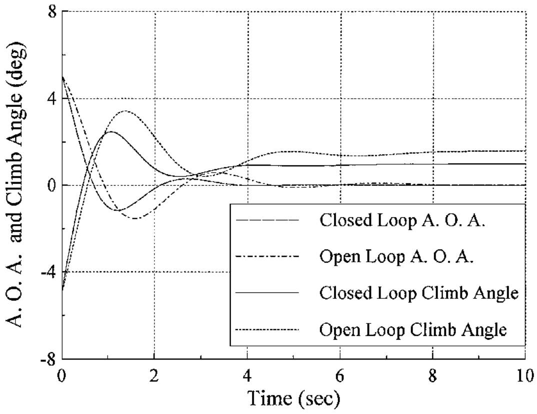

Fig. 3 Initial response of the open-loop and closed-loop climb angles γ and angles of attack α for M_∞ = 0.75.

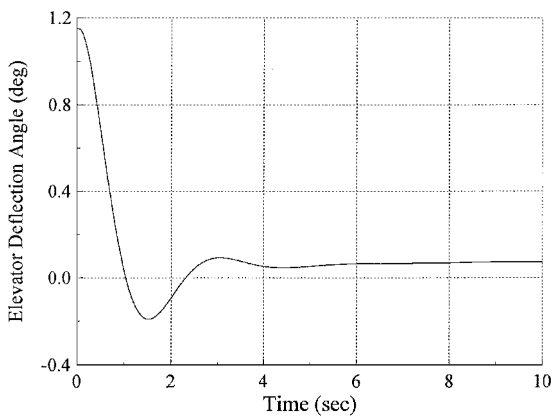

Fig. 4 Time histories of the mixed H_2/H_∞ composite controller δ_e for M_∞ = 0.75.

One can verify that the imaginary parts are not much different for both systems, whereas the magnitude of the real parts of the controlled system are larger than those of the open-loop system. The time history of the control input used in these cases is shown in Fig. 4. It is quite evident that the control surface deflection and its rate computed by the H_2/H_∞ composite controller are quite reasonable and well within reach. Similar solutions were obtained at other Mach numbers as well.

The composite control gains computed for several Mach numbers are shown in Table 2. Note that all Riccati solutions are symmetric, positive semidefinite. The value of γ^* for all four cases is fixed at 30.

B. Comparison with a Pure H_∞ Controller

In this section, a pure H_∞ method applied to the full-order system is compared with the mixed H_2/H_∞ composite controller for the preceding example. The standard H_∞ controller used here is that of Ref. 9. The transfer function of the state model of the aircraft used for the H_∞ controller is from Eq. (11). Performances of the pure H_∞ and the composite H_2/H_∞ controllers are compared in Figs. 5 and 6 for a Mach number of 0.65. The corresponding control time histories are shown in Fig 7. From Figs. 5 and 6, it is evident that for this example the standard H_∞ controller produces acceptable closed-loop performance. However, examination of the results shown here indicates that this H_∞ method results in unreasonably high damping for the fast mode with insufficient damping for the slow mode. The

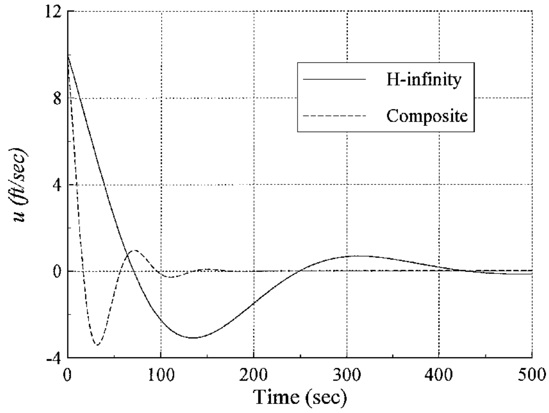

Fig. 5 Horizontal airspeeds using H_∞ and H_2/H_∞ composite controllers for M_∞ = 0.65.

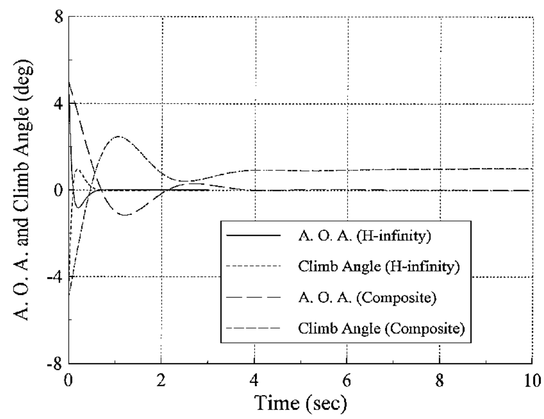

Fig. 6 Climb angles and angles of attack using H_∞ and H_2/H_∞ composite controllers for M_∞ = 0.65.

closed-loop eigenvalues for the preceding system using this H_∞ controller are as follows:

$$\lambda_c = \begin{cases} -10.8434 \pm 4.4891i \\ -0.0084 \pm 0.0176i \end{cases}$$

Comparing these values with those of Table 2, it is evident that the fast mode moves far away to the left of the complex plane, and the slow mode shifts a little closer to the imaginary axis. Furthermore, inspection of Fig. 7 reveals that pure H_∞ control requires a much larger rate of control input than does the H_2/H_∞ technique. This behavior is due to the high gain associated with the H_∞ method. The combination matrices of the control and exogenous matrices, given by the terms in the brackets in the Riccati equations (35) and (36) are not positive. This causes the eigenvalues of the closed-loop system to be shifted apart. This means one set of eigenvalues will shift away from the imaginary axis, while the other set will become closer to it. This problem has been discussed in detail in Ref. 35.

C. Compared to the Real Model

In this section, the mixed H_2/H_∞ composite controller constructed from the estimated model is compared with the same method obtained from the actual aircraft. The aerodynamic coefficients of the actual model of the aircraft for $M_\infty = 0.55$ at altitude 20,000 ft are shown in Table 3 (Ref. 26). Using these data, the coefficients of equation of motion become

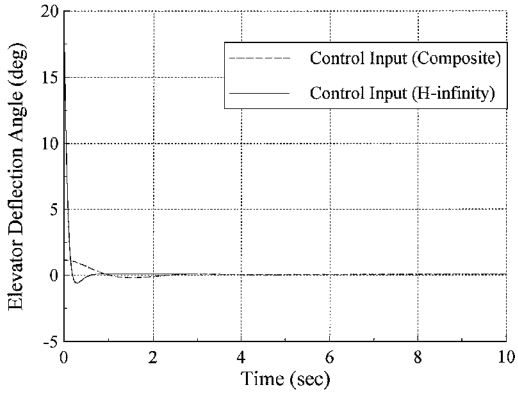

Fig. 7 Time histories of the H_∞ and H_2/H_∞ composite controllers for M_∞ = 0.65.

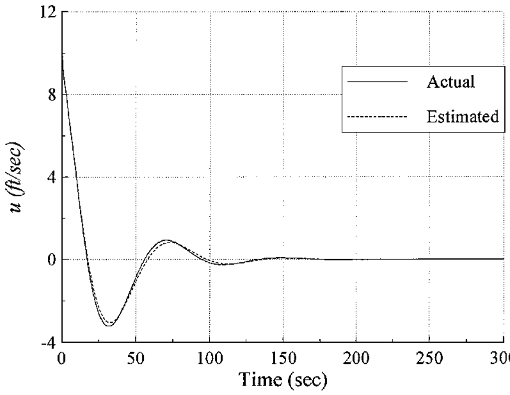

Fig. 8 Estimated and real horizontal airspeed for M_∞ = 0.55, using the estimated and actual mixed H_2/H_∞ composite controllers.

$$A_r = \begin{bmatrix} -0.0066 & -32.1694 & -0.0144 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -0.1114 & -5.8271 & -0.6970 & 543.2224 \\ 0.0006 & -0.4486 & -0.0065 & -1.4398 \end{bmatrix}$$

$$B_{r1} = \begin{bmatrix} -0.0066 & -0.0144 & 0 \\ 0 & 0 & 1 \\ -0.1114 & -0.6970 & 543.2224 \\ 0.0006 & -0.0065 & -1.4398 \end{bmatrix}$$

$$B_{r2} = \begin{bmatrix} 0 \\ 0 \\ -27.7967 \\ -2.8146 \end{bmatrix}$$

$$C_{r1} = [-0.1184 \quad 0 \quad -0.6908 \quad -3.4697]$$

$$C_{r2} = [0 \quad 1 \quad 0 \quad -1], \qquad D_{r12} = -14.0968$$

$$D_{r21} = 1.0, \qquad D_{r11} = D_{r22} = 0$$

where the subscript r refers to the actual model. The responses of the closed-loop systems with the same initial conditions are shown in Figs. 8 and 9, whereas the time history of the control input is presented in Fig. 10. From Figs. 8 and 9, it is obvious that the controller based on estimated coefficients performs as well as the one based on the exact coefficients. The reason for the close agreement between two responses is that the estimated model is very close to the actual model. However, note that unlike for the estimated model, the exact aerodynamic coefficients are not necessarily available at all Mach numbers. This fact can considerably complicate the task of gain scheduling. Furthermore, the uncertainties of the estimated model have been recovered by the H_∞ control method.

Table 3 Coefficients for the actual model at M_∞ = 0.55

Lift coefficients	Drag coefficients	Moment coefficients
$C_{L_1} = 0.460$	$C_{D_1} = 0.021$	$C_{m_1} = 0$
$C_{L_\alpha} = 5.6$	$C_{D_\alpha} = 0.58$	$C_{m_\alpha} = -1.5$
$C_{L_{\dot\alpha}} = 7.5$	$C_{D_{\dot\alpha}} = 0$	$C_{m_{\dot\alpha}} = -1.55$
$C_{L_q} = 8.4$	$C_{D_q} = 0.0$	$C_{m_q} = -23.5$
$C_{L_u} = 0.038$	$C_{D_u} = 0$	$C_{m_u} = 0$
$C_{L_{\delta e}} = 0.405$	$C_{D_{\delta e}} = 0$	$C_{m_{\delta e}} = -1.48$

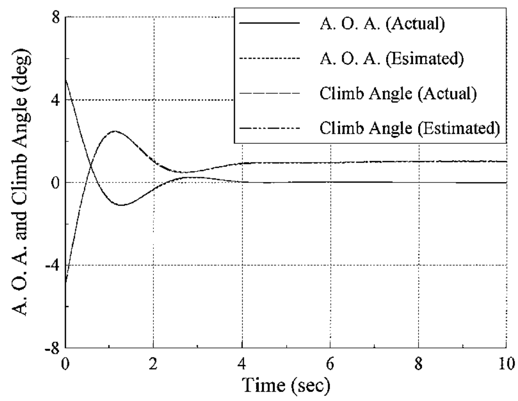

Fig. 9 Climb angles and angles of attack for M_∞ = 0.55, using actual and estimated composite controllers.

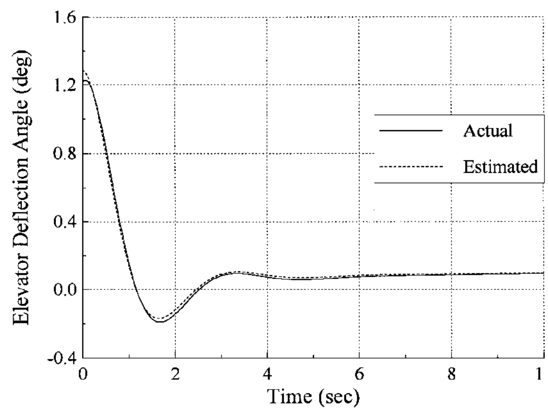

Fig. 10 Time histories of the real and estimated mixed H_2/H_∞ composite controllers for M_∞ = 0.55 using the actual model.

V. Conclusion

The equations for a mixed H_2/H_∞ gain-scheduled controller were formulated. The utility of the method was illustrated by its application to longitudinal equations of motion of an aircraft over a range of subsonic Mach numbers. The singular perturbation method was used to decouple the fast and the slow modes of motion. The H_2/H_∞ optimal control method was applied to the phugoid mode, whereas the H_∞ method was employed for the short period mode. The two schemes were then combined to form the mixed H_2/H_∞ controller. The composite controller was developed based on estimated aerodynamic coefficients of an example aircraft. Differences between the estimates and the actual values were treated as system uncertainties. It was shown that this scheme can provide superior control at flight Mach numbers as high as 0.75.

This control scheme was also compared with one based on the pure H_∞ method with the same value of disturbance attenuation. It was shown that the H_∞ controller can provide acceptable performance either for the fast mode or for the slow mode but not for both simultaneously.

Finally, it was demonstrated that the H_2/H_∞ controller based on the estimated aerodynamic coefficients can recover the system uncertainties. This was illustrated through comparing this controller with one based on the actual aerodynamic coefficients.

In every case, the described method proved capable of controlling both the fast and the slow modes of motion with acceptable inputs.

Appendix: Estimating the Aerodynamic Coefficients and Derivatives

Table A1 Lift and drag aerodynamic coefficients and derivatives

Reference	Equation
26	$C_L = W/\hat{q}S$
26	$C_{L_\alpha} = \left(C_{L_\alpha}\right)_w + \left(C_{L_\alpha}\right)_H$
25 and 26	$(C_{L_\alpha})_w = \dfrac{2\pi \mathrm{AR}}{2 + \sqrt{(\mathrm{AR}^2\beta^2/\kappa^2)\left[1 + \left(\tan^2 \Lambda_{c/2}/\beta^2\right)\right] + 4}}$
25	$C_{Z_\alpha} = -\left(C_{L_\alpha} + C_{D_0}\right)$
28	$C_{Z_u} = -\left(M_\infty \dfrac{\partial C_L}{\partial M_\infty} + 2C_{L_0}\right)$
25	$C_{Z_{\dot{\alpha}}} = -2\left(C_{L_\alpha}\right)_H \eta V_H \dfrac{\partial \varepsilon}{\partial \alpha}$
25	$C_{Z_q} = -2\left(C_{L_\alpha}\right)_H \eta V_H$
26	$(C_T)_{ss} = (C_D)_{ss}$
26	$C_D = C_{D_0} + \dfrac{C_L}{\pi e \mathrm{AR}}$
25	$C_{x_\alpha} = \dfrac{2C_{L_0}}{\pi \mathrm{AR} e} C_{L_\alpha}$
28	$C_{X_u} = -2\left[C_{D_0} + C_{L_0}\tan(\gamma_0)\right] - M_\infty \dfrac{\partial C_D}{\partial M_\infty}$
25	$C_{X_{\dot{\alpha}}}$ and C_{X_q} are negligible

Table A2 Moment and control aerodynamic coefficients and derivatives

Reference	Equation
26	$C_{m_0} = 0$
26	$C_{m_\alpha} = C_{L_{\alpha_w}}\left(X_{\mathrm{cg}} - X_{\mathrm{ac}_w}\right)$
	$-C_{L_{\alpha_H}} \eta_H \dfrac{S_H}{S_w}\left(X_{\mathrm{ac}_H} - X_{\mathrm{cg}}\right)\left(1 - \dfrac{\partial \varepsilon}{\partial \alpha}\right)$
36	$\dfrac{\partial \varepsilon}{\partial \alpha} \approx \dfrac{0.0349 a_{wb}}{\lambda^{0.3}\mathrm{AR}^{0.725}}\left[\dfrac{3\bar{c}}{l_t'}\right]^{0.25}$
26	$C_{m_\alpha} \approx -C_{L_{\alpha_w}}(X_{\mathrm{ac}} - X_{\mathrm{cg}})$
26	$C_{m_{\dot{\alpha}}} = -2C_{L_{\alpha_H}} \eta H \dfrac{S_H}{S_W} \dfrac{X_H}{\bar{c}} \dfrac{\partial \varepsilon}{\partial \alpha}$
25 and 26	$C_{m_q} = -2.2C_{L_{\alpha_H}} \eta H \dfrac{S_H}{S_W} \dfrac{X_H}{\bar{c}}$
26	$C_{m_u} = M_\infty \dfrac{\partial C_m}{\partial M_\infty} = -M_{\infty C_{L_1}} \dfrac{\eta \bar{X}_{\mathrm{ac}}}{\partial M}$
25	$C_{Z_{\delta_e}} = -C_{L_{\alpha_H}} \dfrac{S_H}{S_w}$
25	$C_{m_{\delta_e}} = -C_{L_{\alpha_H}} V_H$
25	$C_{X_{\delta_e}}$ is negligible

References

[1]Kalman, R. E., "When is a Linear System Optimal," *Journal of Basic Engineering*, Vol. 86, March 1964, pp. 1–10.

[2]Kalman, R. E., and Bucy, R. S., "New Results in the Linear Filtering and Prediction Theory," *Journal of Basic Engineering*, Vol. 83, March 1961, pp. 95–108.

[3]Anderson, B. D. O., and Moore, J. B., *Optimal Control Linear Quadratic Methods*, Prentice–Hall, NJ, 1990, pp. 34–51.

[4]Wilson, D. A., and Rubio, J. E., "Computation of Generalized H_2 Optimal Controllers," *International Journal of Control*, Vol. 61, No. 5, 1995, pp. 999–1012.

[5]Zames, G., "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative, Seminorms, and Approximate Inverse," *IEEE Transactions on Automatic Control*, Vol. 26, No. 2, 1981, pp. 301–320.

[6]Francis, B. A., *A Course in H_∞ Controlled Theory*, Springer–Verlag, New York, 1986, pp. 15–42.

[7]Petersen, I. R., "Disturbance Attenuation and H_∞ Optimization: a Design Method Based on the Algebraic Riccati Equation," *IEEE Transactions on Automatic Control*, Vol. 32, No. 5, 1987, pp. 427–429.

[8]Petersen, I. R., "Complete Results for a Class of State Feedback Disturbance Attenuation Problems," *IEEE Transactions on Automatic Control*, Vol. 34, No. 11, 1989, pp. 1196–1199.

[9]Doyle, J. C., Glover, K., Khargonekar, P. P., and Francis, B. A., "State Space Solutions to Standard H_2/H_∞ Control Problems," *IEEE Transactions on Automatic Control*, Vol. 34, No. 8, 1989, pp. 831–847.

[10]Bertein, D. S., and Haddad, W. M., "LQG Control with an H_∞ Bound: a Riccati Equation Approach," *IEEE Transactions on Automatic Control*, Vol. 34, No. 3, 1989, pp. 293–305.

[11]Zhou, K., Glover, K., Bodenheimer, B., and Doyle, J., "Mixed H_2/H_∞ Performance Objectives I: Robust Performance Analysis," *IEEE Transactions on Automatic Control*, Vol. 39, No. 8, 1994, pp. 1564–1574.

[12]Doyle, J., Zhou, K., Glover, K., and Bodenheimer, B., "Mixed H_2/H_∞ Performance Objectives II: Optimal Control," *IEEE Transactions on Automatic Control*, Vol. 39, No. 8, 1994, pp. 1575–1587.

[13]Kaminer, I., Khargonekar, P. P., and Robel, G., "Design of Localizer Capture and Track Modes for a Lateral Autopilot Using H-Infinity Synthesis," *IEEE Control Systems Magazine*, Vol. 10, No. 4, 1990, pp. 13–21.

[14]Yang, J.-S., "H_∞ Robust Control Design for Linear Feedback Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 16, No. 6, 1993, pp. 1131–1137.

[15]Osborne, R. C., Adams, R. J., Hsu, C. S., and Banda, S. S., "Reduced-Order H_∞ Compensator Design for an Aircraft Control Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 2, 1994, pp. 341–345.

[16]Ozbay, H., and Bachmann, G. R., "H_2/H_∞ Controller Design for a Two-Dimensional Thin Airfoil Flutter Suppression," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 4, 1994, pp. 722–728.

[17]Nichols, R. A., Reichert, R. T., and Rugh, W. J., "Gain Scheduling for H-Infinity Controllers: A Flight Control Example," *IEEE Transactions on Control Systems Technology*, Vol. 1, No. 2, 1989, pp. 69–78.

[18]Hyde, R. A., and Glover, K., "The Application of Scheduled H_∞ Controllers to a VSTOL Aircraft," *IEEE Transactions on Automatic Control*, Vol. 38, No. 7, 1993, pp. 1021–1039.

[19]Wang, J. L., "Scheduling of Local Nonlinear Control Laws by Exogenous Signals—An Application to Flight Control," *International Journal of Control*, Vol. 61, No. 4, 1995, pp. 759–782.

[20]Apkarian, P., and Gahinet, P., "A Convex Characterization of Gain-Scheduled H_∞ Controllers," *IEEE Transactions on Automatic Control*, Vol. 40, No. 5, 1995, pp. 853–864.

[21]McRuer, D., Ashkenas, I., and Graham, D., *Aircraft Dynamics and Automatic Control*, Princeton Univ. Press, Princeton, NJ, 1971, pp. 296–414.

[22]Vian, J. L., and Sawan, M. E., "H_∞ Control for a Singular Perturbed Aircraft Model," *Optimal Control Application and Methods*, Vol. 15, No. 2, 1994, pp. 227–289.

[23]Pan, Z., and Basar, T., "H_∞-Optimal Control for Singularly Perturbed Systems, Part I: Perfect State Measurements," *Automatica*, Vol. 29, No. 2, 1993, pp. 401–423.

[24]Pan, Z., and Basar, T., "H_∞-optimal Control for Singularly Perturbed Systems, Part II: Imperfect State Measurements," *IEEE Transaction on Automatic Control*, Vol. 39, No. 9, 1994, pp. 280–299.

[25]Etkin, B., *Dynamics of Flight—Stability and Control*, 2nd ed., Wiley, New York, 1982, pp. 141–199.

[26]Roskam, J., *Airplane Flight Dynamics and Automatic Flight Controls, Part I*, Roskam Aviation and Engineering Corp., Ottawa, KS, 1979, pp. 19–642.

[27]Mclean, D., *Automatic Flight Control Systems*, Prentice–Hall, Englewood Cliffs, NJ, 1990, pp. 17–59.

[28]Nelson, R., *Flight Stability and Automatic Control*, McGraw–Hill, New York, 1989, pp. 83–100.

[29]Prasanath, R. K., and Rotea, M. A., "An Interpolation Approach to Multiobjective H_∞ Design," *International Journal of Control*, Vol. 65, No. 4, 1996, pp. 699–720.

[30]Kokotovic, P., Khalil, H. K., and O'Reilly, J., *Singular Perturbation Method in Control Analysis and Design*, Academic, Orlando, FL, 1986, pp. 94–102.

[31]Xie, L., Fu, M., and de Souza, C. E., "H_∞ Control and Quadratic Stabilization of Systems with Parameter Uncertainty via Output Feedback," *IEEE Transactions on Automatic Control*, Vol. 37, No. 8, 1992, pp. 1253–1256.

[32]Xie, L., and de Souza, C. E., "Robust H_∞ Control for Linear Systems with Norm-Bounded Time Varying Uncertainty," *IEEE Transactions on Automatic Control*, Vol. 37, No. 8, 1992, pp. 1188–1191.

[33]Morgan, L., *Boeing 727 Scrapbook*, 4th printing, Blue Ridge Summit, PA, 1988, pp. 13–71.

[34]*Case Study in Aircraft Design, The Boeing 727*, AIAA, New York, 1978, pp. 1–58.

[35]Shue, S.-P., Sawan, M. E., and Rokhsaz, K., "Limits of Negative Definite and Nondefinite State Weight Matrices for the Second Order Systems," *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Washington, DC, 1995, pp. 1135–1141.

[36]Dommasch, D. O., and Connolly, T. F., *Airplane Aerodynamics*, Pitman, New York, 1967, p. 444.